Expectations – Review
Principle of Maximum Likelihood
Weighted Averages
Linear Least Squares Fitting

Lecture # 6

Physics 2BL

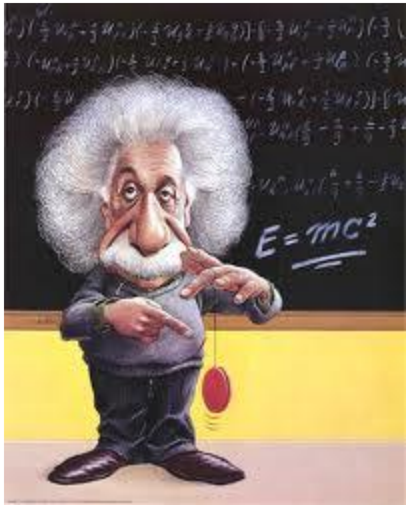
Winter 2011

Outline

- Announcements
- Expectations
- Significant figures
- Principle of maximum likelihood
- Weighted averages
- Least Squares Fitting
- Experiment # 3 analysis
- Brief introduction to Exp. # 4

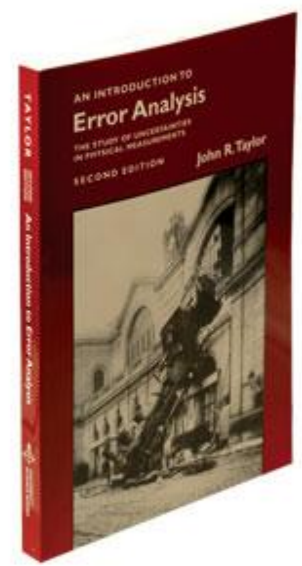
Announcements

1. **Prepare for labs, seek help if needed as resources are available**
2. **Final questions will be extracted from homework and lab concepts**
3. **No lecture next Monday, Feb. 21, university holiday**
4. **Review session on Friday Mar. 4 from 5-7 pm – Ben Heldt running it here (York 2622)**
5. **Final on Monday Mar. 7 during lecture time 7 – 8pm**



Expectations - Review

1. **Understand basic concepts in error analysis**
    a. Significant figures
    b. Propagation of errors – simple forms, general form
    c. Gaussian distributions – mean, standard deviation, standard deviation of the mean
    d. Extract probabilities from t-values
    e. Rejection of data
    f. Weighted averages
    g. Linear least squares



Concepts mentioned in this brief review are not be all inclusive

Expectations - Review

2. **Apply ideas to physics lab situation**
   a. Presentation of measurements and errors using proper number of significant figures
   b. Propagation of errors through calculations (radius and density of earth)
   c. Plot of histograms
   d. Gaussian fits of data – mean, standard deviation, standard deviation of the mean
   a. Extract probabilities from real data – used to determine variation in thickness of racket balls
   b. Testing of a model with measurements – t-score analysis
   c. Design of a voltmeter using physical principles

Significant Figures

What is the correct way to report the following numbers: (Justify your answer)

(a) 653 m $\pm$ 10%

(b) 25.65 $\pm$ $\sqrt{2}$ kg

Principle of Maximum Likelihood

- Best estimates of X and σ from N measurements (x_1 - x_N) are those for which $Prob_{X,\sigma}$ (x_i) is a maximum

The Principle of Maximum Likelihood

Recall the probability density for measurements of some quantity x (distributed as a Gaussian with mean X and standard deviation σ)

$$P_{X,\sigma}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-X)^2}{2\sigma^2}}$$

Normal distribution is one example of $P(x)$.

Now, lets make repeated measurements of x to help reduce our errors.

$$x_1,\ x_2,\ x_3,\ \ldots,\ x_n$$

We define the Likelihood as the product of the probabilities. The larger L, the more likely a set of measurements is.

$$L = P(x_1)P(x_2)P(x_3)\ldots P(x_n)$$

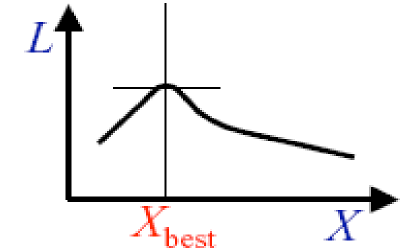
Is L a Probability?

Why does max L give the best estimate?

The best estimate for the parameters of $P(x)$ are those that maximize L.

Using the Principle of Maximum Likelihood:

Prove the mean is best estimate of X



Assume X is a parameter of $P(x)$.
When L is maximum, we must have: $\dfrac{\partial L}{\partial X} = 0$

Lets assume a Normal error distribution and find the formula for the best value for X.

$$L = P(x_1)P(x_2)...P(x_n) = \prod_{i=1}^{n} P(x_i)$$

$$L = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x_i - X)^2}{2\sigma^2}} = \frac{1}{(2\pi)^{\frac{n}{2}}\sigma^n} e^{-\sum_{i=1}^{n}\frac{(x_i - X)^2}{2\sigma^2}}$$

$$L = Ce^{-\chi^2/2}$$

$$\chi^2 = \sum_{i=1}^{n} \frac{(x_i - X)^2}{\sigma^2}$$ Definiinition

$$\frac{\partial L}{\partial X} = 0 = Ce^{-\chi^2/2} \frac{-1}{2} \frac{\partial \chi^2}{\partial X}$$

$$\Rightarrow \quad \frac{\partial \chi^2}{\partial X} = 0$$

$$\frac{\partial \chi^2}{\partial X} = \frac{1}{\sigma^2} \sum_{i=1}^{n} 2(x_i - X) = 0$$

$$\sum_{i=1}^{n} (x_i - X) = 0$$

$$\sum_{i=1}^{n} x_i - nX = 0$$

$$X = \frac{1}{n} \sum_{i=1}^{n} x_i$$

Q.E.D.
the mean

What is the Error on the Mean

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Formula for mean of measurements. (We just proved that this is the best estimate of the true *x*.)

Now, use propagation of errors to get the error on the mean.

$$\sigma_{\bar{x}} = \frac{\partial \bar{x}}{\partial x_1}\sigma_{x_1} \oplus \frac{\partial \bar{x}}{\partial x_2}\sigma_{x_2} \oplus \ldots \oplus \frac{\partial \bar{x}}{\partial x_n}\sigma_{x_n}$$

$$\frac{\partial \bar{x}}{\partial x_i} = \frac{1}{n}$$

$$\sigma_{\bar{x}} = \sqrt{\sum_{i=1}^{n}\left(\frac{\sigma_{x_i}}{n}\right)^2} = \sqrt{n\left(\frac{\sigma}{n}\right)^2} = \frac{\sigma}{\sqrt{n}}$$

What would you do if the x_i had different errors?

We got the error on the mean (SDOM) by propagating errors.

Weighted averages (Chapter 7)

We can use maximum Likelihood (χ^2) to average measurements with different errors.

$$\chi^2 = \sum_{i=1}^{n} \left(\frac{x_i - X}{\sigma_i} \right)^2$$

We derived the result that:

$$\bar{x} = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}$$

Using error propagation, we can determine the error on the weighted mean:

$$\sigma_{\bar{x}} = \frac{1}{\sqrt{\sum_{i=1}^{n} w_i}}$$

What does this give in the limit where all errors are equal?

$$\frac{\partial \chi^2}{\partial X} = 0 = -2 \sum_{i=1}^{n} \frac{x_i - X}{\sigma_i^2}$$

$$\sum_{i=1}^{n} \frac{x_i}{\sigma_i^2} - X \sum_{i=1}^{n} \frac{1}{\sigma_i^2} = 0$$

$$w_i \equiv \frac{1}{\sigma_i^2}$$

$$\sum_{i=1}^{n} w_i x_i = X \sum_{i=1}^{n} w_i$$

$$X = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}$$

Weighted averages

- $X = x = \dfrac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}$ where $w_i = \dfrac{1}{\sigma_i^2}$

- $\sigma_{wav} = \dfrac{1}{\sqrt{\sum_{i=1}^{n} w_i}}$

Example: Weighted Average

Suppose 2 students measure the radius of Neptune.

- Student A gets r=80 Mm with an error of 10 Mm and
- Student B gets r=60 Mm with an error of 3 Mm

What is the best estimate of the true radius?

$$\bar{r} = \frac{w_A r_A + w_B r_B}{w_A + w_B} = \frac{\frac{1}{100}80 + \frac{1}{9}60}{\frac{1}{100} + \frac{1}{9}} = 61.65 \text{ Mm}$$

What does this tell us about the importance of error estimates?

Example:

Compatibility of measurements
Best estimate, Weighted Average

Two measurements of the speed of sound give the answers:

$u_A = 332 \pm 1$ and $u_B = 339 \cdot \pm 3$

(Both in m/s.)

a) How compatable are the measurements? What is the random chance of getting two results that show that difference?
b) What is the best estimate for the speed of sound? What is its uncertainty?

a) To check if the two measurements are consistent, we compute:

$$q = u_A - u_B = 339-332 = 7 \text{ m/s}$$

and: $$\sigma_q = \sqrt{\sigma_{uA}^2 + \sigma_{uB}^2} = 3.16 \text{ m/s}$$

so that: $$t = \frac{q}{\sigma_q} = \frac{339-332}{3.16} = 2.21$$

From Table A we get that 2.21 sigma corresponds to: 97.21%

Therefore the probability to get a worse result is 1-97% ~3%.

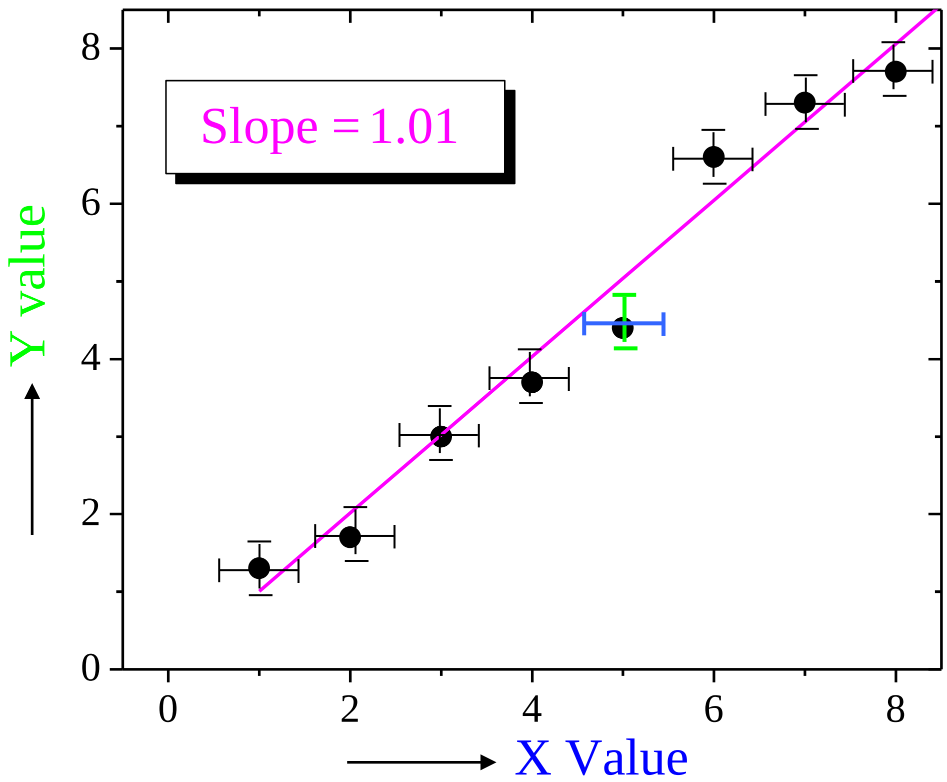
b) Best estimate is the weighted mean:

$$\bar{u} = \frac{w_A u_A + w_B u_B}{w_A + w_B} = \frac{\frac{1}{1}332 + \frac{1}{9}339}{\frac{1}{1} + \frac{1}{9}} = 332.7 \text{ m/s}$$

$$\sigma_{\bar{u}} = \frac{1}{\sqrt{1/w_A + 1/w_B}} = \frac{1}{\sqrt{1/1 + 1/9}} = 0.9 m/s$$

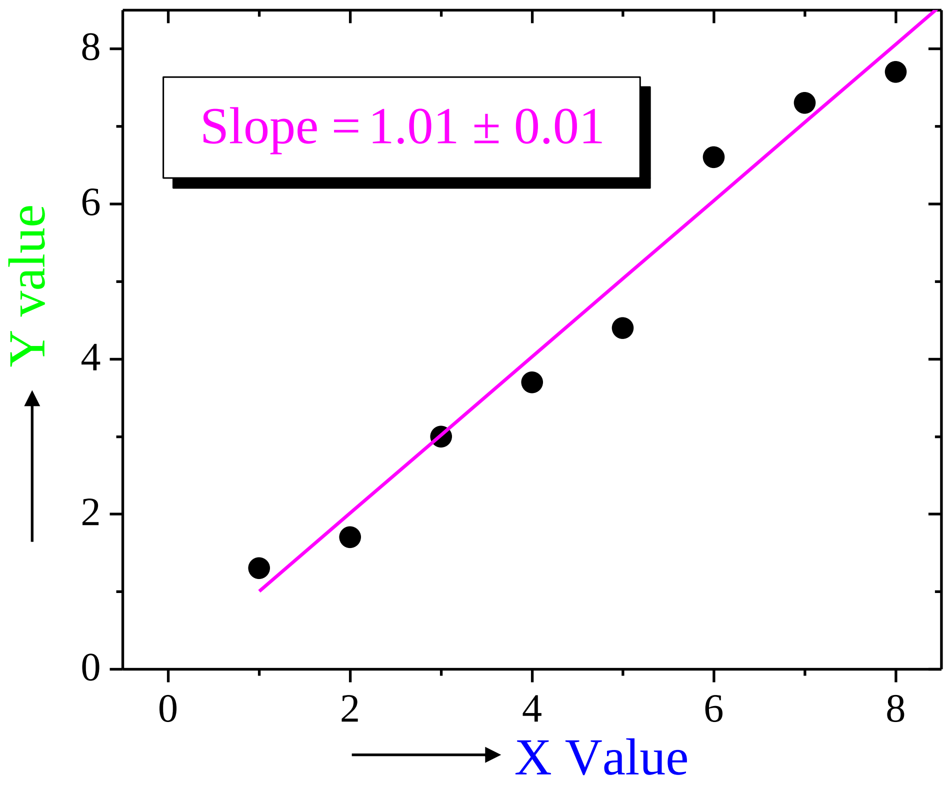
Linear Relationships: $y = A + Bx$ (Chapter 8)

- Data would lie on a straight line, except for errors
- What is 'best' line through the points?
- What is uncertainty in constants?
- How well does the relationship describe the data?



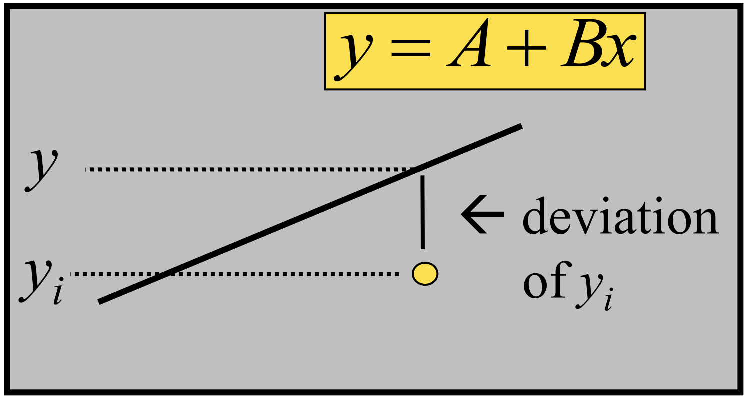
Analytical Fit

- Best means ‘minimize the square of the deviations between line and points’
- Can use error analysis to find constants, error



The Details of How to Do This (Chapter 8)

- Want to find *A, B* that minimize difference between data and line
- Since line above some data, below other, minimize sum of squares of deviations
- Find *A, B* that minimize this sum



$$y_i - y = y_i - A - Bx_i$$

$$\sum_{i=1}^{N}(y_i - A - Bx_i)^2$$

$$\frac{\partial}{\partial A} = \sum y_i - AN - B\sum x_i = 0$$

$$\frac{\partial}{\partial B} = \sum x_i y_i - A\sum x_i + B\sum x_i^2 = 0$$

Finding *A* and *B*

- After minimization, solve equations for *A* and *B*
- Looks nasty, not so bad…
- See Taylor, example 8.1

$$\frac{\partial}{\partial A} = \sum y_i - AN - B\sum x_i = 0$$

$$\frac{\partial}{\partial B} = \sum x_i y_i - A\sum x_i + B\sum x_i^{\,2} = 0$$

$$A = \frac{\sum x_i^2 \sum y_i - \sum x_i \sum x_i y_i}{\Delta}$$

$$B = \frac{N\sum x_i y_i - \sum x_i \sum y_i}{\Delta}$$

$$\Delta = N\sum x_i^{\,2} - \left(\sum x_i\right)^2$$

Uncertainty in Measurements of *y*

- Before, measure several times and take standard deviation as error in *y*
- Can't now, since y_i's are different quantities
- Instead, find standard deviation of deviations

$$\sigma_x = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2}$$

$$\sigma_y = \sqrt{\frac{1}{N-2}\sum_{i=1}^{N}(y_i - A - Bx_i)^2}$$

Uncertainty in *A* and *B*

- *A*, *B* are calculated from x_i, y_i
- Know error in x_i, y_i ; use error propagation to find error in *A*, *B*
- A distant extrapolation will be subject to large uncertainty

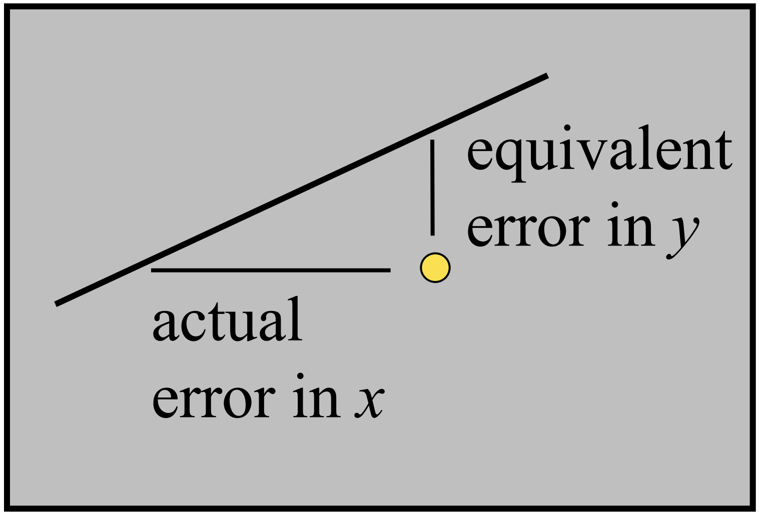
$$\sigma_A = \sigma_y \sqrt{\frac{\sum x_i^2}{\Delta}}$$

$$\sigma_B = \sigma_y \sqrt{\frac{N}{\Delta}}$$

$$\Delta = N \sum x_i^2 - \left(\sum x_i\right)^2$$

Uncertainty in x

- So far, assumed negligible uncertainty in x
- If uncertainty in x, not y, just switch them
- If uncertainty in both, convert error in x to error in y, then add errors



$$\Delta y = B\Delta x$$

$$\sigma_y(equiv) = B\sigma_x$$

$$\sigma_y(equiv) = \sqrt{\sigma_y{}^2 + (B\sigma_x)^2}$$

Other Functions

- Convert to linear
- Can now use least squares fitting to get ln *A* and *B*

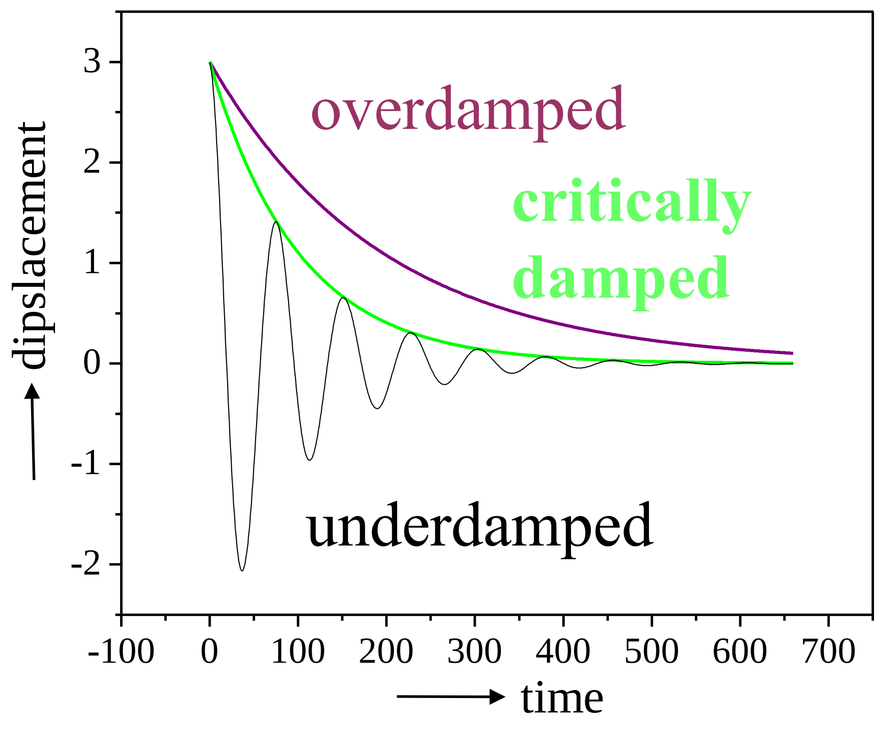
$$y = Ae^{Bx}$$

$$\ln y = \ln A + Bx$$

Experiment 3

- Goals: Test model for damping
- Model of a shock absorber in car
- Procedure: develop and demonstrate critically damped system
- check out setup, take data, do data make sense?
- Write up results - Does model work under all conditions, some conditions? Need modification?

Comparison of the various types of damping



Plotting Graphs

Give each graph a title

Determine independent and dependent variables

Determine boundaries

Include error bars

Demonstrate critical damping: show convincing evidence that critical damping was achieved

- Demonstrate that damping is critical
  - No oscillations (overshoot)
  - Shortest time to return to equilibrium position

Error propagation

(1) $k_{spring} = 4\pi^2 m/T^2$

$\sigma_{kspring} = \varepsilon_{kspring} * k_{spring}$

$\varepsilon_{kspring} = \sqrt{\varepsilon_m{}^2 + (2\varepsilon_T)^2}$

(2) $k_{by\text{-}eye} = m(g\Delta t^*/2\Delta x)^2$

$\sigma_{by\text{-}eye} = \varepsilon_{by\text{-}eye} * k_{by\text{-}eye}$

$\varepsilon_{by\text{-}eye} = \sqrt{(2\varepsilon_{\Delta t^*)}{}^2 + (2\varepsilon_{\Delta x})^2 + \varepsilon_m{}^2}$

The Four Experiments

- **Determine the average density of the earth**

Weigh the Earth, Measure its volume

– Measure simple things like lengths and times

– Learn to estimate and propagate errors

- **Non-Destructive measurements of densities, inner structure of objects**

– Absolute measurements *vs.* Measurements of variability

– Measure moments of inertia

– Use repeated measurements to reduce random errors

- **Construct and tune a shock absorber**

– Adjust performance of a mechanical system

– Demonstrate critical damping of your shock absorber

- **Measure coulomb force and calibrate a voltmeter.**

– Reduce systematic errors in a precise measurement.

Experiment # 4
Outline

- Experiment 4 – electrical forces
- Torsional pendulum
- Review of procedure
- Uncertainties

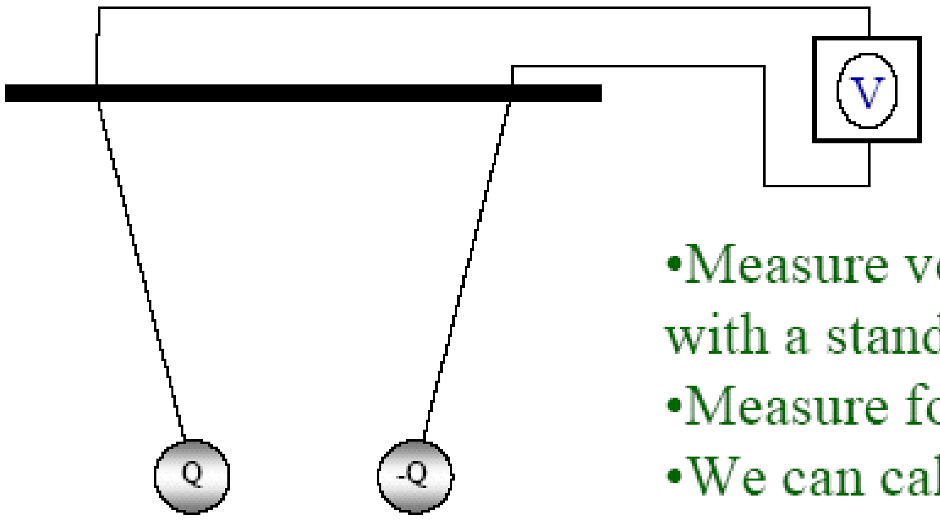
Experiment # 4
Purpose

- Design a means to measure electrical voltage through force exerted on charged object
- Method
- Use Torsional pendulum
- Balance forces, balance torques

Experiment 4
Physics

Construct a device to measure the absolute value of a voltage through the measurement of a force

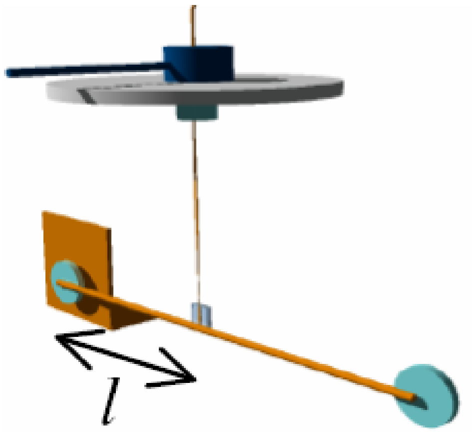
The actual measurements you will make will be of mass, distance, and time but the result will be a measurement of an electric potential in Volts



Measure force and voltage

- Measure voltage difference with a standard meter
- Measure force by deflection
- We can calibrate the voltmeter

Measure κ using Torsional Pendulum



$$F = \kappa\theta / l$$

l - Distance from the suspension to the disk is measured with a ruler

θ - Deflection angle is measured with a protractor

How do we measure the torsion constant κ?

Torsional oscillations $T = 2\pi\sqrt{\frac{I}{\kappa}}$

$$\kappa = \left(\frac{2\pi}{T}\right)^2 I$$

I - Moment of inertia

Remember

- Write-up for Experiment # 3
- Review basic ideas from Taylor chapters 1-9
- Review goals and questions from current and previous labs
- No lecture next Monday, Feb .21, President's Day
- Prepare for Exp. # 4 on electrostatics