PHYSICS 2B - Lecture Notes

Ch. 33: Alternating Current Circuits

Preliminaries

While direct current is conceptually simpler, time-varying or alternating, currents make possible time-varying phenomena that have major practical applications.

Alternating Current

We will be dealing with sources of emf that vary periodically with time, with a period T. Any such function can be represented as a superposition of sinusoidal functions with frequencies that are integer multiples of the frequency

$$f = \frac{1}{T}.$$

So we may confine our attention to potentials of the form

$$V(t) = V_0 \sin(\omega t),$$

where

$$\omega = \frac{2\pi}{T} = 2\pi f .$$

rms Value

The power dissipated in a resistor is proportional to the voltage (or current) squared. Therefore, it makes sense to evaluate the time average of the voltage squared over a full period. Then,

$$\langle V^2 \rangle \equiv \frac{1}{T}\int_0^T V_0^2 \sin^2(\omega t)dt = \frac{V_0^2}{2} \equiv V_{rms}^2 \qquad \Rightarrow \qquad V_{rms} = \frac{V_0}{\sqrt{2}}$$

where the *root-mean-square* voltage gives the average power dissipation over a cycle.

The Power Factor

In an AC circuit, the potential across an element and the current through it both vary with time so that the power dissipated in the element also varies with time. As we shall see, in an AC circuit, there often a phase difference between the potential difference across a circuit element and the current passing through it. We will usually pick the source of emf in the circuit as having a phase of zero so that the phase of the current is relative to the emf. We take the current through the element and the potential difference across it to be

$$V(t) = V_0 \sin(\omega t) \qquad \text{and} \qquad I(t) = I_0 \sin(\omega t + \phi).$$

Then,

$$\langle P \rangle = \langle VI \rangle = \frac{1}{T}\int_0^T V(t)I(t)dt = \frac{V_0 I_0}{T}\int_0^T \sin(\omega t)\sin(\omega t + \phi)dt = \frac{V_0 I_0}{2}\cos\phi = V_{rms} I_{rms}\cos\phi \,.$$

The quantity, $\cos\phi$, is called the power factor.

Phasors

As we have seen the, emf and current in an AC circuit are characterized by a magnitude and a phase. A convenient method of dealing with such quantities takes advantage of properties of complex variables, which we will now review.

Review of Complex Numbers

A complex number can be written as $z = x + iy$, where i is the complex unit defined by

$$i^2 = -1 \qquad \text{or} \qquad \sqrt{-1} = i \,.$$

A complex number can be represented as a vector in a two dimensional plot called an Argand diagram, in which the real part, x , of the complex number is plotted on the horizontal axis and the imaginary part, y, is plotted on the vertical. We can introduce polar coordinates as

$$x = r\cos\phi, \quad y = \sin\phi \qquad \Rightarrow \qquad z = x + iy = r(\cos\phi + i\sin\phi) = re^{i\phi}$$

where

$$r = \sqrt{x^2 + y^2} \qquad \text{and} \qquad \tan\phi = \frac{y}{x} \,.$$

We have used the relation,

$$\cos\phi + i\sin\phi = e^{i\phi} \,,$$

called *Euler's formula.* Two useful results are,

$$\frac{1}{i} = -i \qquad \text{and} \qquad e^{\pm i\pi/2} = \pm\, i$$

Periodic Functions and Phasors

Since any periodic function can be represented as a linear combination of $\cos\omega t$ and $\sin\omega t$, a convenient representation of both functions is, for example,

$$\mathcal{E}(t) = \mathcal{E}_0 e^{i\omega t} \,.$$

Such functions are called *phasors.* A second function, say,

$$I(t) = I_0 e^{i(\omega t + \alpha)} \,,$$

is said to have a phase difference of α with respect to the first function.

Circuit Elements in AC Circuits

We will take the phase of the emf in a circuit to have phase of zero, so that

$$\mathcal{E}(t) = \mathcal{E}_0 e^{i\omega t} .$$

When solving for the current, its phase will be relative to the emf.

Resistors

In a resistor, the relation between the voltage and the current is given by Ohm's law, so that

$$I = \frac{\mathcal{E}}{R} = \frac{\mathcal{E}_0 e^{i\omega t}}{R} = I_0 e^{i\omega t} .$$

Therefore, the phase of the voltage and current are the same and they are said to be in phase.

Capacitor

In a capacitor, $Q = CV$. With V=E, we differentiate with respect to time to get

$$I = \frac{dQ}{dt} = C\frac{dV}{dt} = i\omega C V_0 e^{i\omega t} = \omega C V_0 e^{i(\omega t + \pi/2)} = I_0 e^{i(\omega t + \pi/2)} .$$

Therefore, the current begins its cycle earlier than does the voltage and is said to *lead* the voltage by $\pi/2$ or 90^0. We can write

$$I_0 = \frac{V_0}{X_C} \qquad \text{where} \qquad X_C = \frac{1}{\omega C} \quad .$$

The quantity, X_C, is called the *reactance* of the capacitor.

Inductor

In an inductor, $V = L\frac{dI}{dt}$. With $V(t) = V_0 e^{i\omega t}$, we integrate with respect to time to get

$$I = \frac{V_0}{i\omega L} e^{i\omega t} = \frac{V_0}{\omega L} e^{i(\omega t - \pi/2)} = I_0 e^{i(\omega t - \pi/2)} .$$

Therefore, the current begins its cycle later than does the voltage and is said to *lag* the voltage by $\pi/2$ or 90^0. We can write,

$$I_0 = \frac{V_0}{X_L} \qquad \text{where} \qquad X_L = \omega L \quad .$$

The quantity, X_L, is called the *reactance* of the inductor.

AC Circuits

Tank Circuit

Consider a capacitor with an initial charge Q_0 in an open circuit with an inductor. The switch is closed at $t = 0$. Kirchoff's law gives

$$L\frac{dI}{dt} + \frac{Q}{C} = 0 \qquad \Rightarrow \qquad L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0.$$

This is of the form of the harmonic oscillator equation,

$$\frac{d^2Q}{dt^2} + \omega^2 Q = 0 \qquad \text{where} \qquad \omega^2 = \frac{1}{LC} \quad .$$

A solution may be seen to be $Q = Q_0 e^{i\omega t}$. The charge on the capacitor plates oscillates between positive and negative with angular frequency, ω.

RLC Circuit

Consider a circuit with a resistor, capacitor and inductor in series with an alternating emf. The Kirchoff law in this case is,

$$\mathcal{E} = RI + L\frac{dI}{dt} + \frac{Q}{C}.$$

We differentiate with respect to time, so that

$$\frac{d\mathcal{E}}{dt} = R\frac{dI}{dt} + L\frac{d^2I}{dt^2} + \frac{I}{C}.$$

Taking, $\mathcal{E} = \mathcal{E}_0 e^{i\omega t}$, and seeking a solution of the form $I = I_0 e^{i(\omega t + \alpha)}$, we have

$$\frac{d\mathcal{E}}{dt} = i\omega\mathcal{E}, \qquad \frac{dI}{dt} = i\omega I \qquad \text{and} \qquad \frac{d^2I}{dt^2} = (i\omega)^2 I = -\omega^2 I$$

we have,

$$\mathcal{E} = ZI \qquad \text{where} \qquad Z = R + i\left(\omega L - \frac{1}{\omega C}\right).$$

The Z is called the *impedance* and is may be written,

$$Z = Z_0 e^{i\phi} \qquad \text{where} \qquad Z_0 = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \qquad \text{and} \qquad \tan\phi = \frac{\left(\omega L - \frac{1}{\omega C}\right)}{R}.$$

Dividing by, $e^{i\omega t}$, we have

$$\mathcal{E}_0 = Z_0 I_0 e^{i(\alpha+\phi)} .$$

Since $\mathcal{E}_0$, I_0 and Z_0 are all real, $\alpha + \phi = 0$. Therefore, if ϕ is positive, α is negative and the current will lag the emf.

Finally, we note that Z_0 reaches its minimum value of R when

$$\omega^2 = \frac{1}{LC} .$$

This angular frequency is the *resonant frequency* of the circuit.